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The general warped solution with conical branes in six-dimensional supergravity

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Abstract: We present the general regular warped solution with 4D Minkowski spacetime in six-dimensional gauged supergravity. In this framework, we can easily embed multiple conical branes into the warped geometry by choosing an undetermined holomorphic function. As an example, for the holomorphic function with many zeroes, we find warped solutions with multi-branes and discuss the generalized flux quantization in this case.

Keywords: Classical Theories of Gravity, Supergravity Models.

Contents

1. Introduction

In recent years there has been a lot of interest in models with branes embedded in higher dimensions. One particular motivation is the hope of finding a solution to the notorious cosmological constant problem [1]. From this point of view, six-dimensional (6D) models with codimension-two branes are especially interesting. It was noticed in ref. [2] that the vacuum energy localized on a codimension-two brane results only in a deficit angle of a conical singularity without generating an effective cosmological constant. This observation prompted attempts of realizing the self-tuning idea [3] in the 6D framework.

Specific realizations were first studied in the context of a non-supersymmetric 6D Einstein-Maxwell theory [4, 5]. Background solutions with 4D Minkowski symmetry exist for any values of the brane tensions, once tuning between the gauge flux and a positive bulk cosmological constant is imposed. Embedding such setups in 6D gauged supergravity [6, 7] turns out to be advantageous for several reasons. The so-called Salam-Sezgin supergravity [8] automatically provides the necessary abelian gauge field and positive definite potential. Moreover, the tuning of the gauge flux follows from the equation of motion of a scalar field (the dilaton) also present in that set-up [9]. In fact, for compact extra dimensions, the 4D Minkowski space is a unique solution with maximal symmetry [10]. The study in 6D gauged supergravity was generalized to axially symmetric solutions with a non-trivial warp factor [10–12].

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Unfortunately, in both non-supersymmetric and supersymmetric setups, fine-tuning of the cosmological constant is not completely removed. It was noticed in [5, 9, 10, 13] that taking into account the Dirac quantization condition for the flux results in another constraint, which relates different brane tensions to the discrete (monopole) number characterizing the flux configuration. As a consequence, stable background solutions can exist only for specifically chosen values of the brane tensions. In relation to the question of self-tuning, the cosmology on a codimension-two brane in 6D flux models has been studied in Refs. [14] which come to the same conclusion. It should be noted that this problem can be circumvented in 6D sigma models coupled to gravity. In such framework, the flux quantization constraint is absent and interesting brane solutions with self-tuning features can be found [15].

In this paper we extend the previous studies and present general warped solutions of 6D Salam-Sezgin supergravity. The dependence of the solution on the higher dimensional coordinates is given in terms of an arbitrary holomorphic function $V(z)$. The number of conical singularities, which are free of any curvature divergence, is determined by the zeroes and poles of order $|\alpha| \leq 1$ of $V(z)$. Therefore, if V is single-valued, it has only simple zeroes or poles. For the holomorphic function with a simple zero of $\alpha = 1$, we recover the known warped solutions on S_2 with axial symmetry in extra dimensions [10 − 12]. On the other hand, choosing $V(z)$ with many simple zeroes and/or poles, we can incorporate warped solutions with more than two branes (also without axial symmetry). In this case, however, there also appears a brane with fixed tension. We give an example for the warped solution with multi-branes and derive the 4D Planck mass and the generalized flux quantization in this case.

The paper is organized as follows. After presenting our setup in section 2, we show the general warped solution and analyze its properties in section 3. In section 4, we discuss solutions with two branes in some detail. Then, in section 5, we move to the warped solution with multi-branes. Section 6 contains our conclusions. Finally, a detailed derivation of the general warped solution is presented in the appendix.

2. The model

We consider the relevant[1] bosonic action of the 6D gauged supergravity [6, 7]

$$S_{\rm bulk} = \int d^6X\sqrt{-G}\left[\frac{1}{2}R - \frac{1}{2}(\partial_M\Phi)^2 - \frac{1}{4}e^{-\Phi}F_{MN}F^{MN} - 2g^2\,e^{\Phi}\right], \tag{2.1}$$

supplemented with the 3-brane action:

$$S_{\rm brane} = -\sum_i \int d^4x_i\sqrt{-g_i}\,\Lambda_i \tag{2.2}$$

$$= \int d^6X\sqrt{-G}\,\sum_i \mathcal{L}_{4,i}\,, \tag{2.3}$$

[1]We have dropped the 2-form antisymmetric tensor field from the action, as it does not play any role in our solution.

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where a distributional brane energy density is

$$\mathcal{L}_{4,i} = -\int d^4x_i \sqrt{\frac{-g_i}{-G}}\, \Lambda_i \delta^6(X - X(x_i)). \tag{2.4}$$

Here Λ_i is the tension and $g_{i,\mu\nu}$ is the metric pulled back to the brane worldvolume. The real scalar field Φ is called the dilaton. The $U(1)$ field strength is defined as $F_{MN} = \partial_M A_N - \partial_N A_M$ and the gauge coupling is denoted by g. The 6D fundamental scale has been suppressed, $M_6 = 1$.

The variation of the above action leads to the field equations

$$\partial_M \left(\sqrt{-G}\, e^{-\Phi} F^{MN}\right) = 0, \tag{2.5}$$

$$\frac{1}{\sqrt{-G}}\, \partial_M \left(\sqrt{-G}\, \partial^M \Phi\right) = -\frac{1}{4} e^{-\Phi} F_{MN} F^{MN} + 2g^2 e^{\Phi}, \tag{2.6}$$

and the Einstein equations

$$\begin{aligned} R_{MN} = {} & \partial_M \Phi \partial_N \Phi + g^2 e^{\Phi} G_{MN} + e^{-\Phi}\left(F_{MP} F_N{}^P - \frac{1}{8} G_{MN} F_{PQ} F^{PQ}\right) \\ & + \hat{T}^b_{MN} \end{aligned} \tag{2.7}$$

where $\hat{T}^b_{MN}$ is the brane contribution.

We look for a background solution that is maximally symmetric in four dimensions. We take the general warped ansatz

$$ds^2 = W^2(y) \tilde{g}_{\mu\nu}(x) dx^\mu dx^\nu + \hat{g}_{mn}(y) dy^m dy^n, \tag{2.8}$$

$$F_{mn} = \sqrt{\hat{g}}\, \epsilon_{mn} F(y), \tag{2.9}$$

$$\Phi = \Phi(y), \tag{2.10}$$

and other field components are assumed to have zero vevs. Greek letters $(\mu, \nu = 0, \ldots, 3)$ denote the 4D coordinates, Roman letters $(m, n = 5, 6)$ denote the extra dimensional coordinates, $\tilde{g}_{\mu\nu}(x)$ is a metric of a 4D maximally symmetric spacetime for which $R_{\mu\nu}(\tilde{g}) = 3\lambda \tilde{g}_{\mu\nu}$, and ϵ_{mn} is the Levi-Civita symbol. For this metric ansatz, nonzero Ricci tensor components are given by

$$R_{\mu\nu}(G) = \left(3\lambda - \frac{1}{4} W^{-2} D_m D^m W^4\right) \tilde{g}_{\mu\nu}\,, \tag{2.11}$$

$$R_{mn}(G) = R_{mn}(\hat{g}) - 4W^{-1} D_m D_n W\,, \tag{2.12}$$

and the brane contribution to the Einstein equation is given by

$$\hat{T}^b_{MN} = -\sum_i \frac{\Lambda_i}{\sqrt{\hat{g}}} \left(g_{i,\mu\nu}\, \delta^\mu_M \delta^\nu_N - G_{MN}\right) \delta^2(y - y_i) \tag{2.13}$$

where y_i are brane positions and $g_{i,\mu\nu} = W^2(y_i)\tilde{g}_{\mu\nu}$. Therefore, the field equations and the Einstein equations become

$$\epsilon^{mn}\partial_m(W^4 e^{-\Phi}F(y)) = 0\,, \tag{2.14}$$

$$W^{-4}D_m(W^4 D^m\Phi) = -\left(\frac{1}{2}F^2 e^{-\Phi} - 2g^2 e^{\Phi}\right), \tag{2.15}$$

$$3\lambda - \frac{1}{4}W^{-2}D_m D^m W^4 = -\left(\frac{1}{4}F^2 e^{-\Phi} - g^2 e^{\Phi}\right)W^2\,, \tag{2.16}$$

$$\begin{aligned} R_{mn}(\hat{g}) - 4W^{-1}D_m D_n W &= \partial_m\Phi\partial_n\Phi + \left(\frac{3}{4}F^2 e^{-\Phi} + g^2 e^{\Phi}\right)\hat{g}_{mn} \\ &\quad + \sum_i \frac{1}{\sqrt{\hat{g}}}\Lambda_i\, \hat{g}_{mn}\delta^2(y - y_i)\,. \end{aligned} \tag{2.17}$$

3. The general warped solution

In this section we present the general background solution of the 6D Salam-Sezgin supergravity with maximally symmetric four-dimensions and conical branes. Here we give only the final results, and postpone technical details of the derivation until the appendix.

We parameterize the extra dimensions with a complex coordinate $z = y_5 + iy_6$. As shown in the appendix, the compactness of extra dimensions requires the solution of the form (2.8) only with the flat 4D metric, $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$. The solution is fixed up to an arbitrary holomorphic function $V(z)$ and four real integration constants v, Φ_0, f and ζ_0. The warp factor is given in an implicit form as

$$\frac{\left(W^4(\zeta) - W_-^4\right)^{W_-^4}}{\left(W_+^4 - W^4(\zeta)\right)^{W_+^4}} = \exp\left\{2\gamma^2(W_+^4 - W_-^4)(\zeta - \zeta_0)\right\} \tag{3.1}$$

where

$$\zeta(z) = \frac{1}{2}(\xi + \bar{\xi})\,, \quad \xi = \int^z \frac{d\omega}{V(\omega)}\,, \tag{3.2}$$

and

$$W_\pm^4 = v \pm \sqrt{v^2 - \frac{f^2}{4g^2}}, \quad \gamma^2 = \frac{1}{4}g^2 e^{\Phi_0}\,. \tag{3.3}$$

The metric of the extra dimension is expressed in a conformally flat form as

$$ds_2^2 = e^{2K(z,\bar{z})}dz d\bar{z}\,, \qquad K = \frac{1}{2}\ln\left[\frac{1}{2|V(z)|^2}\frac{P(W)}{W^2}\right] \tag{3.4}$$

with

$$P(W) = \frac{1}{2}\gamma^2 W^{-4}(W_+^4 - W^4)(W^4 - W_-^4). \tag{3.5}$$

The dilaton and the gauge flux are related to the warp factor via

$$\Phi = \Phi_0 - 2\ln W\,, \qquad\qquad F(y) = f e^{\Phi_0}W^{-6}. \tag{3.6}$$

Let us now discuss important properties of the solution. Note first that the warp factor is constrained [2] to lie in the range $W_- \le W \le W_+$. This is the consequence of eq. (3.4) and the fact that the extra dimensions are space-like. Furthermore, reality of the warp factor yields the constraint on the integration constants, $v > 0$, $v^2 > f^2/(4g^2)$. The extrema of the warp factor $W = W_\pm$ correspond to singular points of the metric. From eq. (3.1) we find that $W = W_\pm$ for $\zeta = \pm\infty$.

The allowed form of the holomorphic function is also constrained. Suppose that $V(z)$ at $z = z_i$ is approximated by $V(z) \approx c_i^{-1}(z - z_i)^{\alpha_i}$ with α_i a real number. For $\alpha_i > 1$, ζ and the warp factor are discontinuous around z_i. Moreover, for $\alpha_i < -1$, there exists a curvature singularity at z_i as shown in the appendix. Thus, the order of V in local expansion is restricted to $-1 \le \alpha_i \le 1$. Therefore, we can conclude that a single-valued V can have only simple zeroes or poles.

Let us first consider the single-valued V of the form

$$V(z) \approx c_i^{-1}(z - z_i) \qquad \Rightarrow \qquad \zeta \approx \frac{1}{2} c_i \log |z - z_i|^2 \,, \tag{3.7}$$

where c_i must be real for the warp factor to be single-valued around z_i. In this case, it follows that $\zeta \to \pm\infty$ for $z \to z_i$, so zeroes of $V(z)$ correspond to extrema of the warp factor and singular points of the 2D metric. That is, for $z \to z_i$, $W \to W_+$ if $c_i < 0$, and $W \to W_-$ for $c_i > 0$.

For the simple zeroes of $V(z)$ of the form (3.7) we encounter conical singularities. Indeed, from eqs. (3.1) and (3.4), we find that the 2D metric behaves as

$$K \approx (\beta_\pm - 1) \ln |z - z_i|, \qquad \beta_\pm = \gamma^2 |c_i| W_\pm^{-4} \left(W_+^4 - W_-^4 \right). \tag{3.8}$$

Changing the coordinates to $\rho = |z - z_i|^{\beta_\pm}/\beta_\pm$, $\phi = \mathrm{Arg}(z - z_i)$, the 2D metric near $z = z_i$ becomes $ds_2^2 \approx d\rho^2 + \beta_\pm^2 \rho^2 d\phi^2$. This is a flat 2D metric with a deficit angle $2\pi(1 - \beta_\pm)$. Near $z = z_i$ only eq. (2.17) contains a singular term and it reduces to $\partial\bar{\partial}K = -\frac{\Lambda_i}{2}\delta^2(z - z_i)$. Using the formula $\partial\bar{\partial}\log|z|^2 = 2\pi\delta^2(z)$ we find that the deficit angle should be matched to the brane tensions as

$$\Lambda_i = 2\pi \left[1 - \gamma^2 |c_i| W_\pm^{-4} \left(W_+^4 - W_-^4 \right) \right]. \tag{3.9}$$

We see that the brane tensions are constrained to $\Lambda_i < 2\pi$ (i.e., the deficit angle cannot exceed 2π). If V has more than two simple zeroes, there also appears a brane with fixed tension.

When we have a simple pole of the form $V(z) \sim c_i^{-1}(z - z_i)^{-1}$, the deficit angle around z_i is fixed to -2π so the corresponding brane tension only takes a fixed value. On the other hand, when $V \approx c_i^{-1}(z - z_i)^{\alpha_i}$ with $|\alpha_i| < 1$, a brane with nontrivial tension could be located at z_i. However, since V is not single-valued [3] around z_i, the warp factor would not be well-defined around z_i.

[2] We note that in Romans supergravity [16], the scalar potential is negative definite with $g^2 \to -g^2$ in our setup. In this case, there exists only one real root of $P(W)$ so the value of W is not bounded. Thus, for a finite 4D Planck mass, the space of extra dimensions must be terminated at a 4-brane [11].

[3] We note that the warped solution does not have a symmetry for accommodating a $SU(2)$ monodromy in V, unlike the un-warped case [17].

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The metric solution can be brought to an axially symmetric form, which is a consequence of the existence of a Killing vector for the warped solution [10]. In order to see this we can perform the following locally well-defined change of coordinates:

$$d\zeta = \frac{1}{2}\left(\frac{dz}{V(z)} + \frac{d\bar{z}}{V(\bar{z})}\right), \qquad d\theta = \frac{1}{2i}\left(\frac{dz}{V(z)} - \frac{d\bar{z}}{V(\bar{z})}\right). \tag{3.10}$$

In these new coordinates the metric solution is given by

$$ds^2 = W^2 ds_4^2 + \frac{P(W)}{2W^2}(d\zeta^2 + d\theta^2) \tag{3.11a}$$

$$= W^2 ds_4^2 + \frac{W^4}{2P(W)}dW^2 + \frac{P(W)}{2W^2}d\theta^2 \tag{3.11b}$$

where use is made of eq. (A.18) in the second line. Since the warp factor W is a function of ζ only, the metric becomes independent of the angular variable θ in this local coordinate patch. If the mapping $(z,\bar{z}) \to (\zeta,\theta)$ is single-valued, then the change of coordinates (3.10) is globally well-defined. In this case, we obtain a warped solution with global axial symmetry, which will be discussed in the next section. On the other hand, if the (ζ,θ) coordinates do not cover the whole z plane, the metric does not need to be axially symmetric.

4. Regular warped solutions with axial symmetry

In this section, as a particular solution in our formalism, we will recover the known warped solutions with axial symmetry of extra dimensions in [10–12] and discuss the flux quantization condition. Moreover, we will show a consistent procedure of taking the limit that the warp factor becomes constant.

4.1 The solution

Now let us take a simple ansatz for $V(z)$ as

$$V(z) = \frac{z}{c}, \tag{4.1}$$

with c a complex number. In this case, following the argument in the previous section, there will appear a conical brane with nonzero tension at $z = 0$. Then, the change of coordinates (3.10) become

$$\zeta = \frac{1}{2}\left(c\ln z + \bar{c}\ln\bar{z}\right), \qquad \theta = \frac{1}{2i}\left(c\ln z - \bar{c}\ln\bar{z}\right). \tag{4.2}$$

For the map $\zeta(z,\bar{z})$ to be single-valued around $z = 0$, we note that c must be real [4]. Thus, for this choice of $V(z)$, there exists an axial symmetry of the metric and the change of coordinates is globally well-defined. Since ζ is a log function of $|z|$, there will also appear another conical brane at $z = \infty$.

[4] Also let us take a negative c. Taking a positive c will not change physics.

In this case, by redefining the variable in the metric form (3.11a) or (3.11b) as

$$d\eta = \frac{1}{|c|}\frac{dW}{WP(W)} = \frac{1}{|c|}W^{-4}d\zeta \tag{4.3}$$

and inserting $\psi \equiv (\ln z - \ln \bar{z})/(2i) = -\theta/|c|$ in eq. (4.2), we can find the explicit form of the metric solution in the new coordinate as

$$ds^2 = W^2\eta_{\mu\nu}dx^\mu dx^\nu + a^2W^8d\eta^2 + a^2d\psi^2 \tag{4.4}$$

where

$$W^4 = \frac{1}{2}\left(W_+^4 + W_-^4\right) + \frac{1}{2}\left(W_+^4 - W_-^4\right)\tanh\left[\left(W_+^4 - W_-^4\right)\gamma^2|c|\eta\right], \tag{4.5}$$

$$\begin{aligned} a^2 &= \frac{1}{2}|c|^2\frac{P(W)}{W^2} \\ &= \frac{1}{16}\gamma^2|c|^2\left(W_+^4 - W_-^4\right)^2 W^{-6}\cosh^{-2}\left[\left(W_+^4 - W_-^4\right)\gamma^2|c|\eta\right]. \end{aligned} \tag{4.6}$$

In order to see how the brane tensions are matched in the Einstein equation (A.11), let us consider the asymptotic limits of the metric at two conical singularities. For $\eta \to \pm\infty$, we get the warp factor as

$$W^4 \to W_\pm^4 \mp \left(W_+^4 - W_-^4\right)\exp\left\{\mp 2\left(W_+^4 - W_-^4\right)\gamma^2|c|\eta\right\}, \tag{4.7}$$

as well as

$$a^2 \to \frac{1}{16}\gamma^2|c|^2W_\pm^{-6}\left(W_+^4 - W_-^4\right)^2\exp\left\{\mp 2\left(W_+^4 - W_-^4\right)\gamma^2|c|\eta\right\}. \tag{4.8}$$

Then, let us make a change of coordinate around each singularity by $d\rho_\pm = \mp aW^4d\eta$, which goes to

$$d\rho_\pm = \mp\frac{1}{2}\gamma|c|W_\pm\left(W_+^4 - W_-^4\right)\exp\left\{\mp\left(W_+^4 - W_-^4\right)\gamma^2|c|\eta\right\}d\eta. \tag{4.9}$$

Consequently, the 2d metric goes to

$$ds_2^2 \to d\rho_\pm^2 + \beta_\pm^2\rho_\pm^2d\psi^2, \qquad \beta_\pm \equiv \gamma^2|c|W_\pm^{-4}\left(W_+^4 - W_-^4\right). \tag{4.10}$$

Therefore, we find the brane tensions located at $\eta = \eta_\pm$ to be

$$\Lambda_\pm = 2\pi(1 - \beta_\pm) = 2\pi\left[1 - \gamma^2|c|W_\pm^{-4}\left(W_+^4 - W_-^4\right)\right]. \tag{4.11}$$

Now let us look at the brane conditions more carefully to find the relations between brane tensions explicitly. By eliminating $\frac{W_+^4}{W_-^4}$ in the two brane conditions, we can determine $e^{\Phi_0}|c|$ in terms of the brane tensions as

$$e^{\Phi_0}|c| = 4g^{-2}\frac{2\pi}{\Lambda_+ - \Lambda_-}\left(1 - \frac{\Lambda_+}{2\pi}\right)\left(1 - \frac{\Lambda_-}{2\pi}\right). \tag{4.12}$$

On the other hand, the remaining brane condition gives $\frac{W_+^4}{W_-^4}$ as

$$\frac{W_+^4}{W_-^4} = \frac{2\pi - \Lambda_-}{2\pi - \Lambda_+} \equiv \kappa > 1\,, \tag{4.13}$$

which gives rise to a relation between f and v,

$$\frac{f}{v} = \pm 4g \frac{\sqrt{\kappa}}{(\kappa+1)^2}\,. \tag{4.14}$$

Let us now consider the Planck mass for the warped solution with two branes. Making a dimensional reduction of the 6D Einstein term for the metric form (3.11b), we can read the 4D Planck mass as

$$M_{\rm P}^2 = \frac{1}{2} M_6^4 \int_{W_-}^{W_+} W^3 dW \cdot \int d\theta\,. \tag{4.15}$$

Then, by using eq. (4.2), the Planck mass is shown to be finite as

$$\begin{aligned} M_{\rm P}^2 &= \frac{1}{4}\pi M_6^4 \left(W_+^4 - W_-^4\right) |c| \\ &= \frac{1}{2}\pi M_6^4 v |c| \sqrt{1 - \left(\frac{f}{2gv}\right)^2} \end{aligned} \tag{4.16}$$

where use is made of eq. (3.3) in the second line.

4.2 Flux quantization

Now let us consider the constraint coming from the flux quantization. For compact dimensions, the gauge flux is quantized in the presence of charged particles [5, 9, 10, 13] as

$$\int_{\mathcal{M}_2} F_2 = \frac{2\pi n}{g} \tag{4.17}$$

with $\mathcal{M}_2$ being a 2D compact manifold and n being an integer number. For the metric form (3.11b), the flux is given by

$$F_{W\theta} = \sqrt{\hat{g}}\, \epsilon_{W\theta} f e^{\Phi_0} W^{-6} = \frac{1}{2} \epsilon_{W\theta}\, f e^{\Phi_0} W^{-5}. \tag{4.18}$$

Thus, the quantization condition becomes

$$\frac{1}{8} f e^{\Phi_0} \left(\frac{W_+^4 - W_-^4}{W_+^4 W_-^4}\right) \cdot \int d\theta = \frac{2\pi n}{g}. \tag{4.19}$$

From eqs. (4.19) and (4.2), the flux quantization can be rewritten as

$$\frac{W_+^4 - W_-^4}{W_+^4 W_-^4} f = \frac{8n}{g} \left(e^{\Phi_0} |c|\right)^{-1}. \tag{4.20}$$

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Then, by using eqs. (4.11) and (3.3), we can rewrite the above equation as

$$\frac{\Lambda_+ - \Lambda_-}{2\pi} = \frac{4n^2}{g^2}\left(e^{\Phi_0}|c|\right)^{-1}. \tag{4.21}$$

Therefore, with eq. (4.12), the flux quantization condition only gives a fine-tuning condition between the brane tensions as

$$\left(1 - \frac{\Lambda_+}{2\pi}\right)\left(1 - \frac{\Lambda_-}{2\pi}\right) = n^2. \tag{4.22}$$

Even after the flux quantization, we have fixed only one combination of parameters, $e^{\Phi_0}|c|$ via eq. (4.12) or (4.21), and there is one relation between the parameters f and v from eq. (4.14). However, since c can be absorbed by a rescaling of extra coordinates, η and ψ in eq. (4.4), there appears only one undetermined modulus. However, as will be shown in the next subsection, we find that it is necessary to keep c in the solution explicit to take the limit of the warp factor being constant while maintaining finite values of parameters of the resulting un-warped solution.

4.3 The limit to the un-warped solution

Let us consider a particular limit of the warp solution that the warp factor becomes constant. For this purpose, let us find the consistent limiting procedure of parameters in the solution. From the warped solution (4.4) with eq. (3.3), we can see that the warp factor becomes constant for

$$(W_+^4 - W_-^4) \to 0^+ \quad \text{or} \quad \frac{f}{v} \to \pm 2g. \tag{4.23}$$

In this limit, we take $|c|$ to infinity while keeping the following quantity finite

$$k \equiv |c|\left(W_+^4 - W_-^4\right). \tag{4.24}$$

This limit is also consistent with the relation of the holomorphic function V to the warp factor as in eq. (A.13). This is also necessary for maintaining the finite Planck mass in eq. (4.16) even for $\left(W_+^4 - W_-^4\right) \to 0^+$. Then, in this limit, the warped solution (4.4) becomes

$$ds^2 \to W_+^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{16}\gamma^2 W_+^2 k^2 \left[\frac{d\eta^2 + W_+^{-8} d\psi^2}{\cosh^2(k\gamma^2\eta)}\right]. \tag{4.25}$$

On making a change of coordinate as $d\rho = k\gamma^2 d\eta / \cosh(k\gamma^2\eta)$, the metric can be cast into the following form,

$$\begin{aligned} ds^2 &\to W_+^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{16}\gamma^{-2} W_+^2 \left(d\rho^2 + \beta^2 \sin^2\rho\, d\psi^2\right), \\ \beta &\equiv k\gamma^2 W_+^{-4} = \gamma^2 |c| W_+^{-4}(W_+^4 - W_-^4). \end{aligned} \tag{4.26}$$

Therefore, we get the smooth limit of our warped solution to the un-warped solution where the finite quantity k appears as a deficit angle. In fact, even though the new coordinate

(3.2) does not look well defined for a constant warp factor or $V = 0$, from eqs. (4.2) and (4.3), the change of the original complex coordinate to the final coordinate in eq. (4.4) is well defined for any value of c as

$$d\eta = \frac{1}{2} W^{-4} \left(\frac{dz}{z} + \frac{d\bar{z}}{\bar{z}} \right). \tag{4.27}$$

The resulting un-warped solution describes a sphere with two conical branes with equal tensions,

$$\Lambda_+ = \Lambda_- = 2\pi(1-\beta). \tag{4.28}$$

This result is consistent with the limit of the brane conditions (4.11).

5. Warped solutions with multi-branes

In this section, we generalize the previous warped solution with two branes to the case with more than two branes. Moreover, we find the generalized flux quantization condition and discover new un-warped solutions with multi-branes by taking the limit of the warp factor being constant.

5.1 The solution

Let us take a more general form of holomorphic function V,

$$V(z) = \frac{1}{c} \prod_{i=1}^{N} (z - z_i) \tag{5.1}$$

where c and z_i $(i = 1, \cdots, N)$ are complex numbers. Then, from eq. (3.2), the new coordinate ζ becomes

$$\zeta = \frac{1}{2} \sum_{i=1}^{N} \left(c \int dz \prod_{i=1}^{N} \frac{1}{(z - z_i)} + \text{c.c.} \right). \tag{5.2}$$

By using

$$\prod_{i=1}^{N} \frac{1}{(z - z_i)} = \sum_{i=1}^{N} \left(\prod_{j \neq i} \frac{1}{(z_i - z_j)} \right) \frac{1}{z - z_i}, \tag{5.3}$$

we can rewrite ζ as

$$\zeta = \frac{1}{2} \sum_{i=1}^{N} \left[c \prod_{j \neq i} \frac{1}{(z_i - z_j)} \ln(z - z_i) + \bar{c} \prod_{j \neq i} \frac{1}{(\bar{z}_i - \bar{z}_j)} \ln(\bar{z} - \bar{z}_i) \right]. \tag{5.4}$$

Then, for the single-valuedness of the warp factor $W(\zeta)$ at $z = z_i$ $(i = 1, \cdots, N)$, we must require

$$c \prod_{j \neq i} \frac{1}{(z_i - z_j)} = \bar{c} \prod_{j \neq i} \frac{1}{(\bar{z}_i - \bar{z}_j)} \quad \text{for all } i. \tag{5.5}$$

This means that all the z_i's have to be aligned, $z_i = e^{i\phi}|z_i|$ with a common phase ϕ, while $c = e^{-i(N-1)\phi}|c|$. By a change of coordinates and a redefinition of c, we can take all the z_i's and c to be real. In this case, ζ becomes

$$\zeta = \frac{1}{2}\sum_{i=1}^{N} a_i \ln|z-z_i|^2\,, \qquad a_i \equiv c\prod_{j\neq i}\frac{1}{(z_i - z_j)}\,. \tag{5.6}$$

Now we can see that $z = z_i$ is mapped to $\zeta = +\infty$ $(-\infty)$ for $a_i < 0$ $(a_i > 0)$, finally mapped to $W = W_+(W_-)$. On the other hand, because $\sum_{i=1}^N a_i = 0$, we also find that $z = \infty$ is mapped to $\zeta = 0$.

By following a similar analysis as in the previous section, we can identify the brane tensions located at $z = z_i$ as

$$\Lambda^i_\pm = 2\pi\left[1 - \gamma^2|a_i|W_\pm^{-4}\left(W_+^4 - W_-^4\right)\right], \quad a_i < 0 \;\; (a_i > 0)\,. \tag{5.7}$$

Moreover, we can also find the brane tension located at $z = \infty$ fixed as

$$\Lambda^\infty = 2\pi(2 - N). \tag{5.8}$$

Therefore, the brane tension at $z = \infty$ is fixed by the number of zeroes of $V(z)$, N. We find that eqs. (5.7) and (5.8) for $N = 2$ reproduce the two-brane solution.

Let us see the conditions between unfixed brane tensions explicitly. Suppose that $a_i < 0$ for $i = 1, \cdots, k$ and $a_i > 0$ for $i = k+1, \cdots, N$. Then, summing Λ^i_+ over $i = 1, \cdots, k$ and Λ^i_- over $i = k+1, \cdots, N$ from eq. (5.7), we can get a similar condition as eq. (4.13),

$$\frac{W_+^4}{W_-^4} = \frac{2\pi(N-k) - \sum_{i=k+1}^N \Lambda^i_-}{2\pi k - \sum_{i=1}^k \Lambda^i_+} \equiv \tilde\kappa > 1\,. \tag{5.9}$$

This gives rise to the relation between f and v as

$$\frac{f}{v} = \pm 4g\frac{\sqrt{\tilde\kappa}}{(\tilde\kappa+1)^2}\,. \tag{5.10}$$

Moreover, eliminating $\frac{W_+^4}{W_-^4}$ in the brane condition (5.7), we can determine $e^{\Phi_0}|a_i|$ in terms of the brane tensions: for $i = 1, \cdots, k$,

$$\begin{aligned} e^{\Phi_0}|a_i| = 4g^{-2}&\left(N - 2k + \frac{\sum_{j=1}^k \Lambda^j_+ - \sum_{j=k+1}^N \Lambda^j_-}{2\pi}\right)^{-1} \\ &\times\left(1 - \frac{\Lambda^i_+}{2\pi}\right)\left(N - k - \frac{\sum_{j=k+1}^N \Lambda^j_-}{2\pi}\right), \end{aligned} \tag{5.11}$$

and for $i = k+1, \cdots, N$,

$$\begin{aligned} e^{\Phi_0}|a_i| = 4g^{-2}&\left(N - 2k + \frac{\sum_{j=1}^k \Lambda^j_+ - \sum_{j=k+1}^N \Lambda^j_-}{2\pi}\right)^{-1} \\ &\times\left(1 - \frac{\Lambda^i_-}{2\pi}\right)\left(k - \frac{\sum_{j=1}^k \Lambda^j_+}{2\pi}\right). \end{aligned} \tag{5.12}$$

Thus, from eqs. (5.11) and (5.12), we can see that $\sum_{i=1}^N a_i = 0$ is satisfied. Therefore, from eqs. (5.10), (5.11) and (5.12), the brane conditions only fix the following combinations of parameters: $e^{\Phi_0}|a_i|$ $(i = 1, \cdots, N)$ and f/v. Although two of the brane positions can be always chosen at $z = (0, 1)$ by the invariance of the warp factor, the overall constant of V cannot be absorbed by a rescaling of extra coordinates while keeping the brane positions, unlike the case with two branes. Nonetheless, there appears only one undetermined modulus as in the warped solution with two branes.

Let us now consider the Planck mass for the warped solution with multi-branes. With the metric solution (A.8) with eq. (A.13), a dimensional reduction of the 6D Einstein term to 4D leads to the Planck mass as

$$M_{\rm P}^2 = \frac{1}{8} M_6^4 \int dz d\bar{z}\, \frac{\bar{\partial} W^4}{V}\,. \tag{5.13}$$

Here we note that the divergence theorem in the complex plane is given by

$$\int_R dz d\bar{z}\, (\partial \bar{J} + \bar{\partial} J) = i \oint_{\partial R} (\bar{J} d\bar{z} - J dz) \tag{5.14}$$

with J a complex function and R the integration surface with boundary ∂R. Then, by regarding the surface of compact dimensions as the sum of multiple patches, and applying the divergence theorem, eq. (5.13) becomes

$$M_{\rm P}^2 = \frac{1}{4} \pi M_6^4 \left(\sum_{i=1}^k |a_i| \right) \left(W_+^4 - W_-^4 \right). \tag{5.15}$$

5.2 Generalized flux quantization

Now let us consider the flux quantization condition for the warped solution with multi-branes. For the metric (A.8) with eq. (A.13) in the z complex coordinate, the flux is given by

$$\begin{aligned} F_{z\bar{z}} &= \frac{1}{2} \epsilon_{z\bar{z}} f e^{\Phi_0} W^{-4} e^{2A} \\ &= -\epsilon_{z\bar{z}} f e^{\Phi_0}\, \frac{\bar{\partial}(W^{-4})}{8V}\,. \end{aligned} \tag{5.16}$$

Then, from eq. (4.17), the generalized flux quantization condition is

$$-\frac{1}{8} f e^{\Phi_0} \int dz d\bar{z}\, \frac{\bar{\partial}(W^{-4})}{V} = \frac{2\pi n}{g}. \tag{5.17}$$

Making a similar computation of the complex integral as for the Planck mass in the previous section, we obtain the flux quantization condition as

$$f e^{\Phi_0} \left(\sum_{i=1}^k |a_i| \right) (W_+^{-4} - W_-^{-4}) = \frac{8n}{g}. \tag{5.18}$$

Thus, taking the sum of eq. (5.11), the flux quantization condition corresponds to a generalized version of fine-tuning condition between brane tensions as

$$\left(k - \frac{\sum_{i=1}^k \Lambda_+^i}{2\pi} \right) \left(N - k - \frac{\sum_{i=k+1}^N \Lambda_-^i}{2\pi} \right) = n^2\,. \tag{5.19}$$

5.3 The limit to the un-warped solution

In this section, we generalize the procedure of taking the limit of the warp factor being constant, from the case with two branes to the one with multi-branes.

From the explicit form of the warp factor (A.20), let us define a function χ as

$$e^{W_+^4\chi} \equiv (W^4 - W_-^4)\, e^{2\gamma^2\zeta}\,, \tag{5.20}$$

$$e^{W_-^4\chi} \equiv (W_+^4 - W^4)\, e^{2\gamma^2\zeta} \tag{5.21}$$

where we dropped the integration constant for simplicity. Then, the warp factor can be written as

$$W^4 = \frac{1}{2}(W_+^4 + W_-^4) + \frac{1}{2}\left(W_+^4 - W_-^4\right)\tanh\left[\frac{1}{2}\left(W_+^4 - W_-^4\right)\chi\right] \tag{5.22}$$

with

$$e^{W_+^4\chi} + e^{W_-^4\chi} = \left(W_+^4 - W_-^4\right)\, e^{2\gamma^2\zeta}. \tag{5.23}$$

Now let us take the limit that the warp factor becomes constant. For $\left(W_+^4 - W_-^4\right) \to 0^+$, the general warped solution becomes

$$ds^2 \to W_+^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{16}\frac{\gamma^2 W_+^{-6}\left(W_+^4 - W_-^4\right)^2}{|V|^2\cosh^2\left[W_+^{-4}\left(W_+^4 - W_-^4\right)\gamma^2\zeta\right]}\,dzd\bar{z}\,. \tag{5.24}$$

Then, we can rewrite the resulting un-warped solution as

$$ds^2 \to W_+^2\eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{4}\gamma^{-2}W_+^2\frac{|\partial\omega|^2}{(1+|\omega|^2)^2}\,dzd\bar{z} \tag{5.25}$$

with

$$\omega \equiv \exp\left\{W_+^{-4}\left(W_+^4 - W_-^4\right)\gamma^2\xi\right\}. \tag{5.26}$$

For instance, given the ansatz of V for multi-branes in eq. (5.1), we find the corresponding holomorphic function ω in the un-warped solution as

$$\omega = \prod_{i=1}^N (z - z_i)^{\beta_i}\,, \qquad \beta_i \equiv \gamma^2 a_i W_+^{-4}\left(W_+^4 - W_-^4\right)\,. \tag{5.27}$$

In this case, the appearing brane tensions in the un-warped solution are

$$\begin{aligned}\Lambda^i &= 2\pi\,(1 - |\beta_i|)\,, \qquad i = 1, \ldots, N,\\ \Lambda^\infty &= 2\pi\,(2 - N)\,.\end{aligned} \tag{5.28}$$

Thus, this result is consistent with the limit of the brane conditions, eqs.(5.7) and (5.8). Therefore, for $\left(W_+^4 - W_-^4\right) \to 0^+$ and finite β_i's, we can obtain the smooth limit of the warped solution with multi-branes to the un-warped solution.

From eq. (5.15), in the limit of a constant warp factor, the Planck mass becomes

$$M_{\rm P}^2 \to \frac{1}{4}\pi M_6^4\left(\sum_{i=1}^k |\beta_i|\right)\gamma^{-2} \tag{5.29}$$

which agrees with the result for the un-warped metric with brane tensions given in eq. (5.28) and the 2D curvature of $16\gamma^2$.

6. Conclusion

We have presented the general regular warped solutions with 4D Minkowski spacetime in six-dimensional gauged supergravity. The explicit form of the solution is determined up to an arbitrary holomorphic function which is constrained only for the single-valueness of the warp factor. In this formalism, we have seen that multiple conical branes can be embedded into the warped geometry (also without the axial symmetry of extra dimensions).

For the holomorphic function with one simple zero, we have recovered the known warped solution with two conical branes and discussed the flux quantization in this case. On the other hand, taking a more general form of the holomorphic function with multiple simple zeroes, we found the warped solutions with more than two branes and obtained the Planck mass and the flux quantization condition in an explicit form by complex integrals. In this case, a brane with fixed tension is necessary.

The presence of the undetermined holomorphic function enables us to accommodate the warped solution with extra dimensions of different geometry than S_2. In particular, if $V(z)$ is double periodic the solutions fit in the torus geometry [19]. We leave the detailed discussion on this possibility in a future publication [20].

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A. Derivation of the general warped solution

We consider the equations of motion of the 6D Salam-Sezgin supergravity, eqs. (2.14) to (2.17). Before finding an explicit solution let us study some general consequences of the compactness of extra dimensions and the regularity of the solution. From eqs. (2.15) and (2.16), we get

$$D_m \left(W^4 D^m(\Phi + 2\ln W)\right) = 6\lambda W^2. \tag{A.1}$$

The left-hand side vanishes when integrated over the compact extra dimensions, which implies $\lambda = 0$. Thus, 4D flat space is a unique maximally symmetric solution [10]. Moreover, let us assume that the compact extra dimensions are free of singularities and Φ and W are regular in extra dimensions. Then, by multiplying eq. (A.1) with $\lambda = 0$ by $(\Phi + 2\ln W)$ and integrating by parts, we obtain

$$\Phi = \Phi_0 - 2\ln W \tag{A.2}$$

with Φ_0 being a real constant.

Next, the equation of motion for the gauge field (2.14) immediately leads to the solution for the field strength:

$$F(y) = fe^{\Phi}W^{-4} = fe^{\Phi_0}W^{-6} \tag{A.3}$$

with f being a real constant.

Now let us turn to the Einstein equations. Using $R_{mn}(\hat{g}) = K(y)\hat{g}_{mn}$ with K being the Gauss curvature and taking the relations (A.2) and (A.3), the R_{mn} equation (2.17) becomes

$$\begin{aligned} K\hat{g}_{mn} - 2W^{-2}D_mD_nW^2 &= e^{\Phi_0}\left(\frac{3}{4}f^2W^{-10} + g^2W^{-2}\right)\hat{g}_{mn} \\ &\quad + \sum_i \frac{1}{\sqrt{\hat{g}}}\Lambda_i\,\hat{g}_{mn}\,\delta^2(y-y_i)\,. \end{aligned} \tag{A.4}$$

Therefore, the trace-free part gives

$$D_mD_nW^2 = \frac{1}{2}D^2W^2\,\hat{g}_{mn}. \tag{A.5}$$

This implies the existence of a Killing vector [10],

$$V_m = \sqrt{\hat{g}}\,\epsilon_{mn}D^nW^2 \tag{A.6}$$

satisfying $D_{(m}V_{n)} = 0$.

By integrating the trace of eq. (A.4) over the extra dimensions, we get the Euler number as

$$\chi = \frac{1}{2\pi}\int d^2y\sqrt{\hat{g}}\left[W^{-2}D^2W^2 + e^{\Phi_0}\Big(\frac{3}{4}f^2W^{-10} + g^2W^{-2}\Big)\right] + \frac{1}{2\pi}\sum_i\Lambda_i. \tag{A.7}$$

Now let us introduce a complex coordinate $z = y_5 + iy_6$ and take the following ansatz for the metric:

$$ds^2 = W^2(z,\bar{z})\left(\eta_{\mu\nu}dx^\mu dx^\nu + e^{2A(z,\bar{z})}dzd\bar{z}\right) \tag{A.8}$$

where $\eta_{\mu\nu}$ is the 4D flat metric. Inserting the above ansatz into (2.16) and (2.17) with eqs. (A.2) and (A.3), the $R_{\mu\nu}$, R_{zz}, and $R_{z\bar{z}}$ equations become, respectively,

$$-4e^{-2A}(4\partial B\bar{\partial}B + \partial\bar{\partial}B) = e^{\Phi_0}\left(g^2 - \frac{1}{4}f^2e^{-8B}\right), \tag{A.9}$$

$$-4\partial^2B + 4(\partial B)^2 + 8\partial B\partial A = 4(\partial B)^2, \tag{A.10}$$

$$\begin{aligned} -6\partial\bar{\partial}B - 4\partial B\bar{\partial}B - 2\partial\bar{\partial}A &= 4\partial B\bar{\partial}B + \frac{1}{2}e^{2A}e^{\Phi_0}\left(g^2 + \frac{3}{4}f^2e^{-8B}\right) \\ &\quad + \sum_i\Lambda_i\delta^2(z-z_i) \end{aligned} \tag{A.11}$$

with $B \equiv \ln W$, $\partial \equiv \frac{\partial}{\partial z}$ and $\bar{\partial} \equiv \frac{\partial}{\partial\bar{z}}$.

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In order to find a solution to the bulk equations we first rewrite eq. (A.10) as

$$e^{2A}\,\bar\partial(e^{-2A}\bar\partial B)=0. \tag{A.12}$$

This can be easily solved up to an arbitrary holomorphic function $V(z)$,

$$V(z)=e^{-2A}\bar\partial B. \tag{A.13}$$

For constant B (i.e. $V=0$), from eq. (A.9) we find the condition $f^2=4g^2$. This choice corresponds to un-warped solutions that were considered in ref. [9] and (with multiple branes) in ref. [17]. Here we will focus on the warped solution. For $V\neq 0$ we can multiply both sides of eq. (A.9) by V and integrate over $\bar z$ to get

$$V\partial e^{4B}=-\frac{1}{4}e^{\Phi_0}g^2\left(e^{4B}+\frac{f^2}{4g^2}e^{-4B}-2v(z)\right) \tag{A.14}$$

with $v(z)$ being a holomorphic function. Let us now introduce a new holomorphic variable[5]

$$\xi=\int^z\frac{d\omega}{V(\omega)}. \tag{A.15}$$

Then, we can rewrite the equation (A.14) as

$$\frac{\partial}{\partial\xi}W^4=-\frac{1}{4}e^{\Phi_0}g^2\left(W^4+\frac{f^2}{4g^2}W^{-4}-2v(\xi)\right). \tag{A.16}$$

Combining the reality of the warp factor W and the holomorphicity of $v(\xi)$, we find that W is a function of real part of ξ only and v is a real constant. That is, for the real variable

$$d\zeta=\frac{1}{2}d\left(\xi+\bar\xi\right)=\frac{1}{2}\left(\frac{dz}{V(z)}+\text{c.c.}\right), \tag{A.17}$$

eq. (A.16) becomes

$$f(W)dW\equiv\frac{W^3}{P(W)}dW=d\zeta \tag{A.18}$$

where

$$P(W)=\frac{1}{2}\gamma^2\left(-W^4-u^2W^{-4}+2v\right),\quad \gamma^2\equiv\frac{1}{4}e^{\Phi_0}g^2,\quad u^2\equiv\frac{f^2}{4g^2}. \tag{A.19}$$

Therefore, we find that the equation of motion is reduced to the simple ordinary differential equation (A.18). This can be integrated easily to yield

$$\frac{\left(W^4(\zeta)-W_-^4\right)^{W_-^4}}{\left(W_+^4-W^4(\zeta)\right)^{W_+^4}}=\exp\left\{2\left(W_+^4-W_-^4\right)\gamma^2(\zeta-\zeta_0)\right\} \tag{A.20}$$

[5]There was a similar approach to finding a warped solution in 6D Einstein gravity with a nonzero bulk cosmological constant [18]. In their case, only 4D dS space solution is allowed.

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with ζ_0 an integration constant and

$$W_\pm^4 = v \pm \sqrt{v^2 - u^2}\,. \tag{A.21}$$

One can also show that this solution satisfies the remaining equation (A.11) away from the branes. This completes the derivation of the general solution, eqs. (3.1) to (3.6).

Before closing this section, let us comment on the possible curvature singularities of our general warped solutions. The 6D Ricci scalar for our general warped solutions is given as

$$R = 4\gamma^2 W^{-6}(5W^4 - u^2 W^{-4} + 2v). \tag{A.22}$$

Thus, the 6D Ricci scalar is finite, as W is constrained in the range $W_- \leq W \leq W_+$. Furthermore, let us also consider a higher curvature invariant in the bulk such as

$$R_{MNPQ}R^{MNPQ} = -24W^{-4}(DW)^4 - 16W^{-2}(D_m D_n W)^2 + 4K^2 \tag{A.23}$$

with the Gaussian curvature from eq. (A.4) as

$$K = W^{-2}D^2W^2 + e^{\Phi_0}\left(\frac{3}{4}f^2W^{-10} + g^2W^{-2}\right). \tag{A.24}$$

By plugging into eq. (A.23) the following quantities

$$(DW)^2 = 2W^{-4}P(W), \tag{A.25}$$

$$D^2W = 2W^{-5}(WP'(W) - 3P(W)), \tag{A.26}$$

$$(D_m D_n W)^2 = W^{-10}\left[8\left|-3\frac{V}{\bar{V}}P + 2W^4\bar{\partial}V\right|^2 + 2(WP'(W) - 3P(W))^2\right], \tag{A.27}$$

we can see that $\bar{\partial}V$ must be finite for a regular higher curvature invariant. Certainly, $\bar{\partial}V = 0$ for any regular point of V. At singularities, however, $\bar{\partial}V$ can be different from zero. Suppose V has an expansion $V \sim (z - z_i)^{\alpha_i}$ around z_i with α_i a real number. We note that for $\alpha_i \neq -1$,

$$\begin{aligned}\bar{\partial}(z - z_i)^{\alpha_i} &= -\alpha_i(z - z_i)^{\alpha_i+1}\underbrace{\bar{\partial}\frac{1}{z}}_{\bar{\partial}\partial\ln|z|^2} + \bar{\partial}(z - z_i)^{\alpha_i}\Big|_{\mathrm{BC}} \\ &= -2\pi\alpha_i(z - z_i)^{\alpha_i+1}\delta^2(z - z_i) \\ &\quad + \frac{i}{2}|z - z_i|^{\alpha_i-1}e^{-i\,\mathrm{Arg}(z-z_i)}(1 - e^{2\pi i\alpha_i})\,\delta\left(\mathrm{Arg}(z - z_i)\right)\end{aligned} \tag{A.28}$$

where BC is the branch cut for z_i with non-integer α_i. Thus we can distinguish several cases:

- For natural number α_i, $\bar{\partial}V$ vanishes.
- For non-integer number satisfying $\alpha_i > 1$ or $-1 < \alpha_i < 1$, there is a 1D delta function singularity along the branch cut.

- For $\alpha_i = -1$, there is a 2D delta function singularity.
- For $\alpha_i < -1$, there is a curvature singularity.

For $\alpha_i > 1$, ζ and hence the warp factor is discontinuous around z_i. The delta funtion singularities appearing in the second and third cases are as singular as the conical singularities, in the sense that they can be regularized at the level of higher order curvature expansion. Therefore, in the end, we have to restrict ourselves to $|\alpha_i| \leq 1$.

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